Chapter 5

Invariance Signatures

The aim of any invariant pattern recognition technique is to obtain a representation of the pattern in a form that is invariant under some specified transformations of the original image. Ideally, such a technique would produce a representation of the pattern which was not only invariant, but which also uniquely characterized the input pattern.

This goal is not necessarily realisable, nor is it always as desirable as it might seem. In many pattern recognition problems, the aim is to produce a system which classifies input patterns as belonging to a particular *class*, rather than to identify uniquely every single input pattern presented. In such cases a unique representation for each and every possible input pattern can actually be a disadvantage. What is required is an invariant representation that retains enough information for distinct classes to be distinguished. It is by no means necessary that members of the same class be distinguishable in the invariant representation used.

Invariant pattern recognition has a long history, as was discussed in Chapter 2. One approach is to study the ways in which *local* features change under the action of *global* transformations of the image. Such considerations lead naturally to the study of Lie transformation groups, and many, varied invariant pattern recognition techniques make use of the Lie theory [Hoffman, 1966; Hoffman, 1978; Ferraro and Caelli, 1988; Cole, Murase and Naito, 1991; Rubinstein et al., 1991; Segman et al., 1992; Squire and Caelli, 1995].

In this chapter the theory of Lie transformation groups will be developed, and their significance for invariant pattern recognition made clear. A new shift-, rotation- and scale- invariant function of a two-dimensional contour with respect to a given Lie transformation group, the *Invariance Measure Density Function*, will then be derived. It will be shown that several such functions can be combined to yield an *Invariance Signature* for the contour. This Invariance Signature has several properties that make it attractive for implementation in a MBNN: it is based on local properties of the contour, so initial calculations are inherently parallel; it is statistical in nature, and its resolution can be chosen at the designer's discretion, allowing direct control over the dimensionality of the network implementation. However, the use of the Invariance Signature is by no means limited to neural network implementations. Whilst the Invariance Signature is not "strongly-invariant" in the sense discussed in §2.1 (patterns are not uniquely represented), it will be shown in Chapter 7 that *classes* of patterns are represented sufficiently differently for optical character recognition applications.

5.1 Lie Transformation Groups and Invariance

5.1.1 Definition of a Group

A group is a set of elements and an associated operation which maps elements of the group into each other. Consider a set of elements G and an associated operation $\oplus$. Let the elements of G be denoted by $\epsilon_1, \epsilon_2, \dots$. For G to be a group, it must have the following properties:

1. Closure:

$$\forall \ \epsilon_i, \epsilon_j \in G, \ \ \epsilon_i \oplus \epsilon_j \in G \tag{5.1}$$

2. Associativity:

$$\forall \ i, j, k, \ \ (\epsilon_i \oplus \epsilon_j) \oplus \epsilon_k = \epsilon_i \oplus (\epsilon_j \oplus \epsilon_k) \tag{5.2}$$

3. Identity Element I:

$$\exists \ I : \ \forall \ \epsilon \in G, \ \ \epsilon \oplus I = \epsilon \tag{5.3}$$

4. Inverse Elements:

$$\forall \ \epsilon \in G, \ \exists \ \epsilon^{-1} : \ \ \epsilon \oplus \epsilon^{-1} = I \tag{5.4}$$

5.1.2 One Parameter Lie Groups in Two Dimensions

An important class of groups for pattern recognition is the Lie transformation groups. A Lie group is a continuous transformation group with a differentiable structure. For two-dimensional invariant pattern recognition, the most interesting groups are the one-parameter Lie transformation groups defined on the plane. These transformation groups include rotation, dilation and translation: transformations with respect to which invariance is often desired in pattern recognition systems. These are smooth transformations of the form

$$\begin{aligned} x' &= \mu(x, y, \alpha) \\ y' &= \nu(x, y, \alpha). \end{aligned} \tag{5.5}$$

The parameter α determines which element of the group the transformation is. For instance, if α_0 corresponds to the identity element, we have

$$\begin{aligned} x' &= \mu(x, y, \alpha_0) = x \\ y' &= \nu(x, y, \alpha_0) = y. \end{aligned} \tag{5.6}$$

There is a vector field $\vec{g} = \begin{bmatrix} g_x & g_y \end{bmatrix}^{\mathrm{T}}$ associated with each Lie group G. This vector field gives the direction in which a point (x, y) is "dragged" by an infinitesimal transformation under the action of the group. It is given by

$$\begin{aligned} g_x(x, y) &= \left.\frac{\partial \mu}{\partial \alpha}\right|_{\alpha=\alpha_0} \\ g_y(x, y) &= \left.\frac{\partial \nu}{\partial \alpha}\right|_{\alpha=\alpha_0}. \end{aligned} \tag{5.7}$$

This vector field $\vec{g}$ allows an operator $\mathscr{L}_G$ to be defined,

$$\mathscr{L}_G = g_x \frac{\partial}{\partial x} + g_y \frac{\partial}{\partial y}. \tag{5.8}$$

The operator $\mathscr{L}_G$ is called the *generator* of the transformation group G. It is called the generator of the group because it can be used to construct the finite transformation corresponding to the infinitesimal dragging described in Equations 5.7. This is done by "summing" the repeated applications of the infinitesimal transformation.

5.1.3 From Infinitesimal to Finite Transformations

Let us consider the case in which we know the direction of the vector field specifying the infinitesimal transformation at each point, as given by Equations 5.7, and we wish to construct the Equations 5.5 specifying the finite transformation. We will consider the transformation of x in detail. For a small change in the group parameter from the identity element, $\alpha = \alpha_0 + \Delta\alpha$, we can approximate the change in x by

$$x' = x + \Delta x \approx x + \Delta\alpha \left.\frac{\partial \mu}{\partial \alpha}\right|_{\alpha=\alpha_0} \tag{5.9}$$

We now wish to find a finite transformation corresponding to n applications of the $\Delta\alpha$ transformation. This will approximate the finite transformation corresponding to the group element specified by parameter $\alpha = n\Delta\alpha$. Let x_i correspond to the value of x' after i applications of the transformation specified by $\Delta\alpha$. We obtain

$$\begin{aligned} x_0 &= x \\ x_1 &= x_0 + \frac{\alpha}{n}\mathscr{L}_G x_0 = \left(1 + \frac{\alpha}{n}\mathscr{L}_G\right) x \\ x_2 &= x_1 + \frac{\alpha}{n}\mathscr{L}_G x_1 \\ &= \left(1 + \frac{\alpha}{n}\mathscr{L}_G\right) x_1 \\ &= \left(1 + \frac{\alpha}{n}\mathscr{L}_G\right)^2 x \end{aligned} \tag{5.10}$$

and thus

$$x_n = \left(1 + \frac{\alpha}{n}\mathscr{L}_G\right)^n x.$$

In the limit as $n \to \infty$, the approximation becomes exact, and the finite transformation is given by

$$\mu(x, y, \alpha) = \lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\mathscr{L}_G\right)^n x. \tag{5.11}$$

Similarly,

$$\nu(x, y, \alpha) = \lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\mathscr{L}_G\right)^n y. \tag{5.12}$$

5.1.4 Derivation of the Rotation Transformation

As an example, we will derive the rotation transformation from the knowledge only that each point P should be dragged in a direction at right angles to the line from the origin to P by the infinitesimal transformation, as shown in Figure 5.1. We will denote the rotation group R. The parameter of the group is θ. From Figure 5.1, we see that

$$\begin{aligned} dx &= -r\,d\theta \sin\phi \\ &= -\sqrt{x^2+y^2}\,d\theta \frac{y}{\sqrt{x^2+y^2}} \\ &= -y\,d\theta. \end{aligned} \tag{5.13}$$

Similarly,

$$dy = x\,d\theta. \tag{5.14}$$

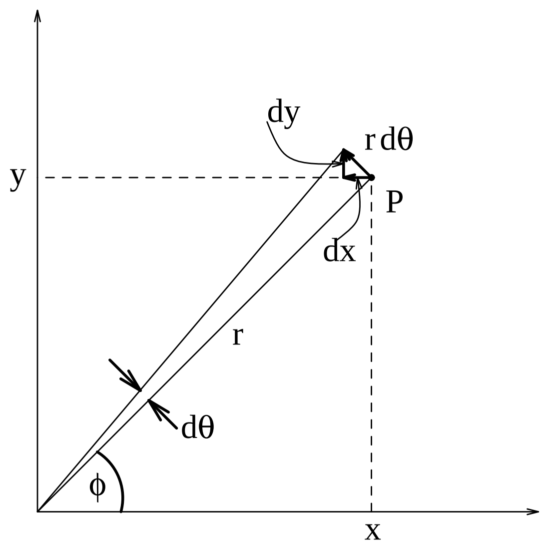


Figure 5.1: Infinitesimal transformation leading to rotation.

The generator of the rotation group R is thus

$$\mathscr{L}_R = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{5.15}$$

Consequently, the corresponding finite transformation for x is

$$\mu(x,y,\theta) = \lim_{n\to\infty}\left(1 + \frac{\theta}{n}\mathscr{L}_R\right)^n x. \tag{5.16}$$

Using the binomial expansion,

$$\begin{aligned}\mu(x,y,\theta) = \lim_{n\to\infty}\Bigg[1 + n\frac{\theta}{n}\mathscr{L}_R + \frac{n(n-1)}{2!}\frac{\theta^2}{n^2}\mathscr{L}_R^2 + \frac{n(n-1)(n-2)}{3!}\frac{\theta^3}{n^3}\mathscr{L}_R^3 \\ + \frac{n(n-1)(n-2)(n-3)}{4!}\frac{\theta^4}{n^4}\mathscr{L}_R^4 + \cdots\Bigg]x.\end{aligned} \tag{5.17}$$

We note that

$$\lim_{n\to\infty}\frac{n!}{(n-k)!n^k} = 1, \tag{5.18}$$

and

$$\begin{aligned}\mathscr{L}_R x &= -y \\ \mathscr{L}_R^2 x &= -x \\ \mathscr{L}_R^3 x &= y \\ \mathscr{L}_R^4 x &= x,\end{aligned} \tag{5.19}$$

after which the values repeat. We can therefore write

$$\begin{aligned}\mu(x,y,\theta) &= \left[x - \theta y - \frac{\theta^2}{2!}x + \frac{\theta^3}{3!}y + \frac{\theta^4}{4!}x + \cdots\right] \\ &= x\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) - y\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= x\cos\theta - y\sin\theta.\end{aligned} \tag{5.20}$$

Similarly,

$$\nu(x,y,\theta) = x\sin\theta + y\cos\theta. \tag{5.21}$$

These are the well-known equations describing a rotational transformation by an angle θ. The procedure above can be used to generate the finite transformation corresponding to any Lie operator.

5.1.5 Functions Invariant Under Lie Transformations

A function is said to be invariant under the action of a transformation if all points of the function are mapped into other points of the function by the action of the transformation on the coordinate system. Such functions are important in pattern recognition, since it is often desirable to recognize patterns independently of common image transformations such as rotation, dilation and scaling.

Consider a function $F(x,y)$. We wish to determine its invariance with respect to a Lie transformation group G. Let

$$\vec{g}(x,y) = \begin{bmatrix} g_x(x,y) & g_y(x,y) \end{bmatrix}^{\mathrm{T}} \tag{5.22}$$

be the vector field corresponding to the generator of the Lie group, $\mathscr{L}_G$. F is constant with respect to the action of the generator if

$$\mathscr{L}_G F = 0. \tag{5.23}$$

This can be written in terms of the vector field as

$$\nabla F \cdot \vec{g}(x,y) = 0, \tag{5.24}$$

that is

$$\frac{\partial F}{\partial x}g_x + \frac{\partial F}{\partial y}g_y = 0, \tag{5.25}$$

or

$$\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{g_y}{g_x}. \tag{5.26}$$

Now consider a contour C parameterized by t on which F is constant,

$$\forall t \;\; F(x(t), y(t)) = K. \tag{5.27}$$

Since F is constant on the contour, we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = 0, \tag{5.28}$$

and consequently

$$\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{dy}{dx}. \tag{5.29}$$

Thus the condition for invariance with respect to the Lie transformation group generated by $\mathscr{L}_G$, given in Equation 5.24, holds if

$$\frac{dy}{dx} = \frac{g_y}{g_x}. \tag{5.30}$$

on the contour.

The condition derived in Equation 5.30 has a very natural interpretation. It says that a contour is invariant under the action of a group G if the tangent to the contour at each point is in the same direction as the vector field $\vec{g}$ corresponding to the infinitesimal transformation that generates the group.

5.2 From Local Invariance Measures to Global Invariance

We now propose a new shift-, rotation- and dilation-invariant signature for contours. We call this an *Invariance Signature*, since it is derived from the degree to which a given contour is consistent with invariance under a set of Lie transformation groups.

5.2.1 The Local Measure of Consistency

We have seen in Equation 5.30 that in order for a contour C to be invariant under a transformation group G the tangent to the contour must be everywhere parallel to the vector field defined by the generator of the group. We now define the *Local Measure of Consistency* with invariance under a transformation group G at a point (x, y) on C,

$\iota_G(x, y)$.

$$\iota_G(x,y) = \left|\hat{\theta}(x,y) \cdot \hat{g}_G(x,y)\right| \tag{5.31}$$

The absolute value is used because only the orientation of the tangent vector is significant, not the direction. At each point both the tangent vector to the contour, $\vec{\theta}(x, y)$ and the vector field $\vec{g_G}(x, y)$ are normalized:

$$\hat{g}_G(x,y) = \frac{g_x(x,y)\hat{\imath} + g_y(x,y)\hat{\jmath}}{\sqrt[+]{g_x^2(x,y) + g_y^2(x,y)}} \tag{5.32}$$

and

$$\hat{\theta}(x,y) = \frac{\hat{\imath} + \frac{dy}{dx}\hat{\jmath}}{\sqrt[+]{1 + \left(\frac{dy}{dx}\right)^2}}, \tag{5.33}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in the x and y directions respectively. Substituting Equations 5.32 and 5.33 in Equation 5.31, we obtain

$$\iota_G(x,y) = \frac{1}{\sqrt[+]{1 + \left[\frac{g_y(x,y) - g_x(x,y)\frac{dy}{dx}}{g_x(x,y) + g_y(x,y)\frac{dy}{dx}}\right]^2}} \tag{5.34}$$

5.2.2 The Invariance Measure Density Function

Equation 5.31 is a mapping $C \mapsto [0, 1]$, which gives a measure of the degree to which the tangent at each point is consistent with invariance under G. We wish to find a function which characterizes the degree to which the entire contour C is consistent with invariance under G. Such a function is the density function for the value of ι_G in $[0, 1]$, $I(\iota_G)$, which we will call the *Invariance Measure Density Function*. The more points from C that are mapped to values close to 1 by Equation 5.31, the more consistent C is with invariance under G. The shape of $I(\iota_G)$ is a descriptor of C, and we will show that $I(\iota_G)$ is invariant under rotations and dilations of C (see Theorem 5.1). Translation invariance is obtained if the origin of coordinates in which ι_G is calculated is chosen to be the centroid of C.

It is interesting to note that there is evidence from psychophysical experiments that a measure of the *degree of invariance* of a pattern with respect to the similarity group of transformations (rotations, translations and dilations) is important in human pattern recognition [Caelli and Dodwell, 1984]. The measure proposed here might be seen as a mathematical formalization of this notion. Moreover, its implementation in a neural network architecture is consistent with Caelli and Dodwell's (1984, p. 159) statement of a proposal due to Hoffman [1966; 1978]:

> Hoffman's fundamental postulate was that the coding of orientation at various positions of the retinotopic map by the visual system, discovered by Hubel and Wiesel (1962) and others, actually provides the visual system with "vector field" information. That is, the visual system, on detecting specific orientation and position states ("Φ/P codes"), spontaneously extracts the path curves (interpreted as visual contours) of which the local vectors are tangential elements.

First, however, we must establish the form of $I(\iota_G)$. Without loss of generality, consider C to be parameterized by $t : t \in [t_0, T]$. The arc length s along C to a given value of the parameter t is:

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\left.\frac{dx}{dt}\right|_{\tau}\right)^2 + \left(\left.\frac{dy}{dt}\right|_{\tau}\right)^2} \, d\tau. \tag{5.35}$$

The total length of C is thus $S = s(T) = \oint_C ds$. For well-behaved functions $F(x,y)$, we can construct $s(t)$ such that we can reparameterize C in terms of s. Thus we can rewrite Equation 5.31 to give ι_G in terms of s. For simplicity, we will first consider the case in which ι_G is a monotonic function of s, as shown in Figure 5.2.

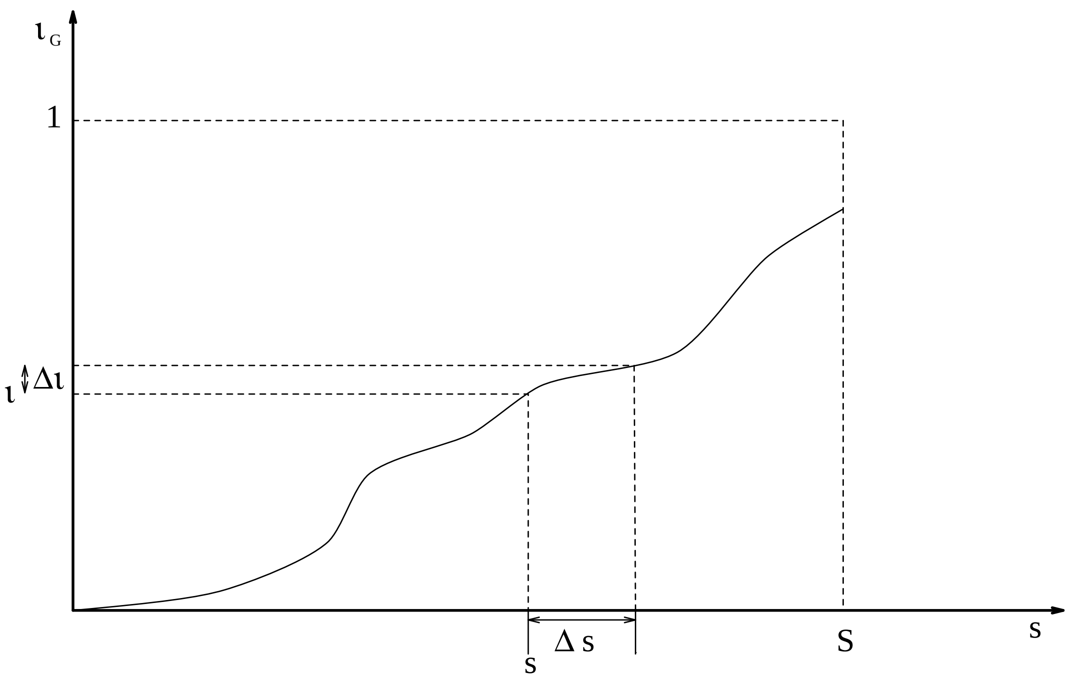


Figure 5.2: Local Measure of Consistency as a function of arc length.

The Invariance Measure Density is:

$$\begin{aligned} I\left(\iota_G\right) &= \lim_{\Delta\iota_G \to 0} \left| \frac{\Delta s}{S \Delta\iota_G} \right| \\ &= \frac{1}{S} \left| \frac{ds}{d\iota_G} \right| . \end{aligned} \tag{5.36}$$

$I\left(\iota_G\right)$ can be interpreted as the probability density function for ι_G at points $(x(s), y(s))$, where s is a random variable uniformly distributed on $[0, S]$. It is clear that for the general case, in which ι_G is not a monotonic function of s, the function could be broken up into piecewise monotonic intervals, and the contribution to the density from each interval summed. The general form for a specific value $\iota_G{}'$ is therefore

$$I\left(\iota_G{}'\right) = \frac{1}{S} \sum_{s \in [0,S]: \iota_G(s) = \iota_G{}'} \left| \frac{ds}{d\iota_G} \right| . \tag{5.37}$$

Theorem 5.1 *The Invariance Measure Density, $I\left(\iota_G\right)$, is invariant under translations, rotations and dilations of the contour C with respect to which it is calculated.*

Proof That $I\left(\iota_G\right)$, defined in Equation 5.37, is invariant under translations of the contour C is trivial, since, as defined in §5.2.2, ι_G is calculated with the origin at the centroid of the contour C. Rotation and dilation invariance can be proved by construction. Since the transformations for rotation and dilation in the plane commute,[1] we can consider a two-parameter Abelian group corresponding to a rotation by an angle θ and a dilation by a positive factor β. The coordinates x and y are transformed according to

$$\begin{aligned} x' &= \beta\left(x \cos\theta - y \sin\theta\right) \\ y' &= \beta\left(x \sin\theta + y \cos\theta\right) . \end{aligned} \tag{5.38}$$

We are interested in the relationship between the arc length function $s(t)$ defined in Equation 5.35 for the original parameterized contour $(x(t), y(t))$ and the new function $s'(t)$ when the coordinates are transformed according to Equation 5.38. We find that

$$\begin{aligned} \frac{dx'}{dt} &= \beta \cos\theta \frac{dx}{dt} - \beta \sin\theta \frac{dy}{dt} \\ \frac{dy'}{dt} &= \beta \sin\theta \frac{dx}{dt} + \beta \cos\theta \frac{dy}{dt} . \end{aligned} \tag{5.39}$$

[1] As a consequence of Clairaut's theorem, they commute when the functions to which the transformations are applied are both defined and continuous in a neighbourhood around each point of the function.

Combining these, we find that

$$\begin{aligned}\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)^2 =& \beta^2\left[\cos^2\theta\left(\frac{dx}{dt}\right)^2-2\cos\theta\sin\theta\frac{dx}{dt}\frac{dy}{dt}+\sin^2\theta\left(\frac{dy}{dt}\right)^2\right.\\ &\left.+\sin^2\theta\left(\frac{dx}{dt}\right)^2+2\cos\theta\sin\theta\frac{dx}{dt}\frac{dy}{dt}+\cos^2\theta\left(\frac{dy}{dt}\right)^2\right]\\ =&\beta^2\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right].\end{aligned} \tag{5.40}$$

This result can be substituted into Equation 5.35 to obtain

$$s'(t)=\beta s(t). \tag{5.41}$$

This clearly indicates that the total arc length is $S'=\beta S$, and the derivative of $s(t)$ is also scaled. Substituting into Equation 5.37, we obtain

$$\begin{aligned}I'\left(\iota_G{}'\right) &= \frac{1}{S'}\sum_{\iota_G(s)=\iota_G{}'}\left|\frac{ds'}{d\iota_G}\right|\\ &= \frac{1}{\beta S}\sum_{\iota_G(s)=\iota_G{}'}\beta\left|\frac{ds}{d\iota_G}\right|\\ &= \frac{1}{S}\sum_{\iota_G(s)=\iota_G{}'}\left|\frac{ds}{d\iota_G}\right|\\ &= I\left(\iota_G{}'\right)\\ &\quad Q.E.D.\end{aligned} \tag{5.42}$$

Thus we have demonstrated that $I\left(\iota_G{}'\right)$ is invariant under rotations and dilations of the contour C.

Invariance Measure Densities For Specific Contours

In order to demonstrate how the Invariance Measure Density defined above can be applied, we will evaluate $I\left(\iota_G\right)$ for a specific contour.

The Square Let the contour C be a square of side $2L$ centred at the origin, as shown in Figure 5.3. We will find the Invariance Measure Density for C with respect to rotation, $I_{C_{rot}}\left(\iota\right)$. By symmetry, we need only find $I_{C_{rot}}$ for one side of the square. On the side indicated by the dashed line in Figure 5.3, x and y can be expressed in

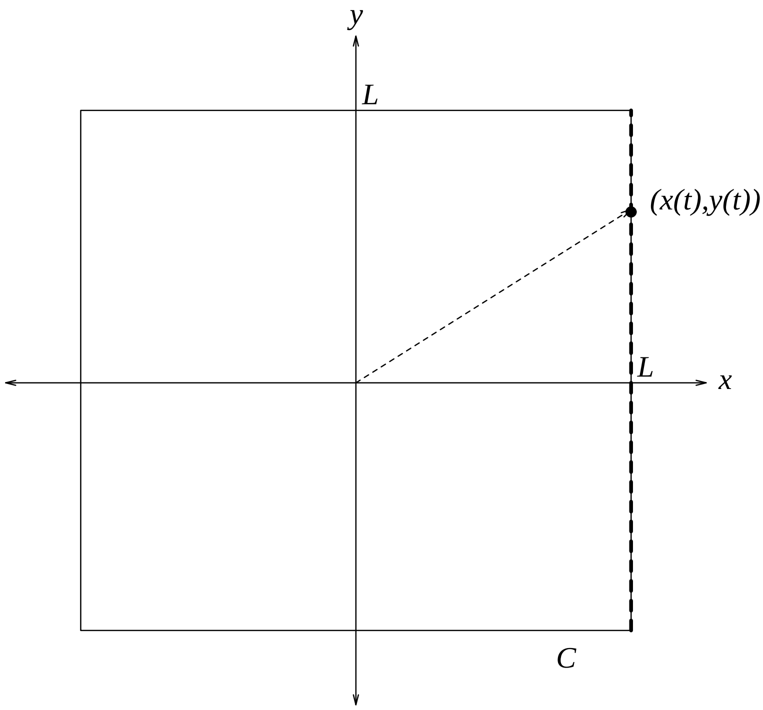


Figure 5.3: Square of side 2L.

terms of a parameter t as

$$\begin{aligned} x &= L \\ y &= t, \qquad -L \leq t \leq L. \end{aligned} \tag{5.43}$$

Here the arc length $s(t) = t + L$, and the total is $S = 2L$. If $\dot{x}(t)$ and $\dot{y}(t)$ are the derivatives of x and y with respect to t, Equation 5.34 can be rewritten as

$$\iota_{C_{rot}}(s) = \frac{1}{\sqrt{1 + \left[\frac{g_y(s)\dot{x}(s) - g_x(s)\dot{y}(s)}{g_x(s)\dot{x}(s) + g_y(s)\dot{y}(s)}\right]^2}} \tag{5.44}$$

Here we have $\dot{x}(t) = 0$ and $\dot{y}(t) = 1$. Equations 5.15 and 5.43 can be substituted into Equation 5.44 to give

$$\iota_{C_{rot}}(s) = \frac{1}{\sqrt{1 + \left(\frac{s-L}{L}\right)^2}}, \tag{5.45}$$

which can be inverted to yield

$$s = L\left(1 + \sqrt{\frac{1}{\iota^2_{C_{rot}}} - 1}\right); \tag{5.46}$$

differentiating,

$$\frac{ds}{d\iota_{C_{rot}}} = \frac{-L}{\iota_{C_{rot}}^2 \left(1 - \iota_{C_{rot}}^2\right)}. \tag{5.47}$$

Using Equation 5.37, we arrive at our final result:

$$\begin{aligned} I_{C_{rot}}\left(\iota_{C_{rot}}\right) &= \frac{1}{2L} \sum_{\iota'_{C_{rot}}=\iota_{C_{rot}}} \left|\frac{ds}{d\iota_{C_{rot}}}\right|_{\iota'_{C_{rot}}} \\ &= \frac{1}{2L} \times 2 \times \frac{L}{\iota_{C_{rot}}^2 \left(1 - \iota_{C_{rot}}^2\right)} \\ &= \frac{1}{\iota_{C_{rot}}^2 \left(1 - \iota_{C_{rot}}^2\right)}, \qquad \iota_{C_{rot}} \in \left[\frac{1}{\sqrt{2}}, 1\right]. \end{aligned} \tag{5.48}$$

Note that, as required, the scale of the square L does not appear in this result. The factor of 2 arises because $\iota_{C_{rot}}$ is a symmetric function of s, so the sum has two terms. This function, shown in Figure 5.4, is characteristic of the square.

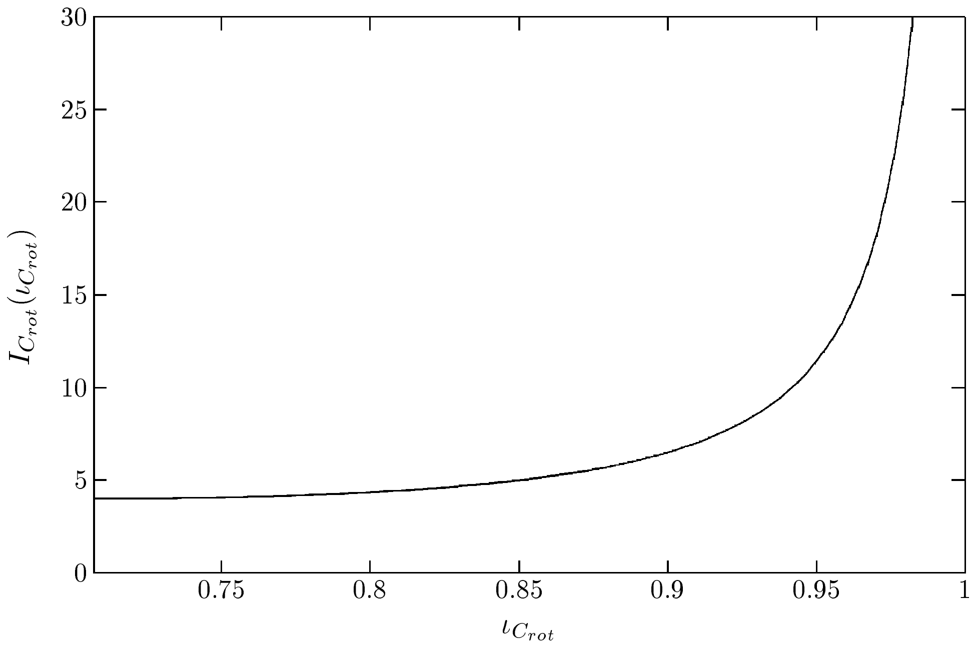


Figure 5.4: Invariance Density Measure with respect to rotation for a square.

5.2.3 Invariance Space: Combining Invariance Measure Densities

We have seen how to compute the Invariance Measure Density Function for a contour C with respect to invariance under a Lie transformation group G. We now consider the case in which the Invariance Measure Density Function is calculated with respect to a number of groups, and the results combined to provide a more complete characterization of the transformational properties of a contour C. This operation can be considered to be a mapping of each point from the two dimensional image space to the interior of a unit hypercube in an n-dimensional *invariance space*, where each of the n dimensions corresponds to a particular sort of invariance. Equation 5.49 shows this for the case of a three-dimensional invariance space, where the dimensions correspond to the Local Measure of Consistency ι with respect to the transformations of rotation, dilation and translation:

$$(x, y) \mapsto \begin{bmatrix} \iota_{rot} & \iota_{dil} & \iota_{trans} \end{bmatrix}^{\mathrm{T}}. \tag{5.49}$$

The distribution of points in this invariance space is characteristic of the contour C. This particular three-dimensional invariance space is the one that will be used for the experimental application of Invariance Signatures. Since each of the component invariance measure densities is invariant, this n-dimensional Invariance Signature is invariant under rotations, dilations, translations and reflections of the input image.

Vector Fields Corresponding to Rotation, Dilation and Translation Groups

The vector fields for the generators of the transformation groups for rotation, dilation and translation are given in normalized form. All can be derived using Equation 5.7: for rotation invariance

$$\vec{g}_{rot}(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} -y & x \end{bmatrix}^{\mathrm{T}}, \tag{5.50}$$

and for dilation invariance

$$\vec{g}_{dil}(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}. \tag{5.51}$$

The translation invariance case is somewhat different. In fact, the term "translation invariance" is something of a misnomer. What is in fact measured in this case is the degree to which the contour is "linear". The vector field used is constant for all (x, y), and the direction is determined by finding the eigenvector corresponding to the dominant eigenvalue, $\vec{e}_1$, of the coordinate covariance matrix of all the points in the contour.[2] The direction of this eigenvector is the principal direction of the contour.

To place this in the same context as the previous two cases, the "translation invariance" factor is a measure of the degree to which the tangents at each point in the contour are consistent with a function invariant under translation in the direction of the vector $\vec{e}_1$. Since $\vec{e}_1$ is calculated from the image each time it is required, this measure is invariant under rotations, dilations and translations of the image. The vector field

[2] $\vec{e}_1$ is chosen to be a unit vector.

for the translation invariance case is thus:

$$\vec{g}_{trans}(x, y) = \begin{bmatrix} e_{1x} & e_{1y} \end{bmatrix}^{\mathrm{T}} \tag{5.52}$$

Uniqueness

It should be noted that this representation of the image is not unique, unlike some previous integral transform representations [Ferraro and Caelli, 1994]. The combination of individual Invariance Measure Densities into an Invariance Space does, however, increase the discriminant properties. As an example, removing two opposite sides of a square will not alter its rotation and dilation Invariance Signatures, but it will change the translation Invariance Signature. Likewise, a single straight line has the same translation Invariance Signature as any number of parallel straight lines, however they are spaced. The rotation and dilation Invariance Signatures, however, are sensitive to these changes.

5.2.4 Discrete Invariance Signatures

For any computer application of Invariance Signatures, a discrete version must be developed. The natural choice is the frequency histogram of the value of ι_G. For a continuous contour, this is obtained by dividing the interval $[0, 1]$ into n "bins" and integrating $I(\iota_G)$ over each bin. For bins numbered from b_0 to b_{n-1}, the value in bin k is thus

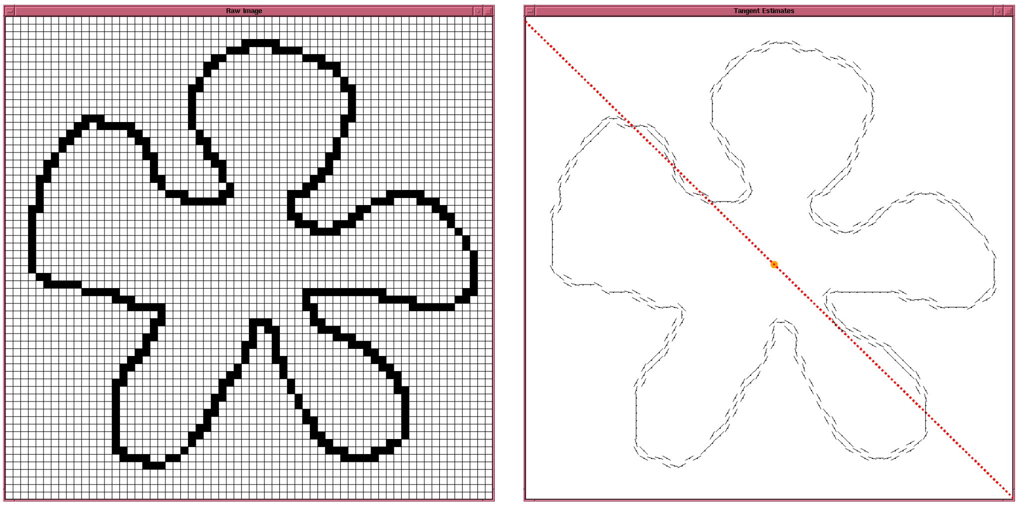
$$b_k = \int_{\frac{k}{n}}^{\frac{k+1}{n}} I(\iota_G) d\iota_G. \tag{5.53}$$

Since $I(\iota_G)$ is a probability density function, the sum of the values of the bins must be one.

For a system using images of sampled contours, a true frequency histogram of the estimated local measures of consistency may be used. The designer of a system using these Invariance Signatures must choose the number of bins n into which the data is grouped. It will be seen in Chapter 7 that this choice is not arbitrary for real systems.

An example of a sampled contour and the estimated tangent vectors at each point is shown in Figure 5.5. The circle indicates the centroid of the contour, and the dashed line shows the direction of $\vec{e}_1$. The estimated discrete Invariance Signatures are shown in Figure 5.6, for 20 bins.

It would be expected that this "flower"-shaped contour would have Invariance Signatures which reflect a quite strong dilation-invariant component corresponding to the approximately radial edges of the "petals", and also a significant rotation-invariant component due to the ends of the petals which are approximately tangential to the



1. Original Contour   2. Tangent Estimates

Figure 5.5: Example of a sampled contour and its estimated tangents.

1. Rotation   2. Dilation   3. Translation

Figure 5.6: 20 bin discrete Invariance Signatures for the contour in Figure 5.5.

radial edges. This is indeed what is observed in Figure 5.6.